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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM

No. 1070

INTEGRAL METHODS IN THE THEORY OF THE BOUNDARY LAYER

By L. G. Loitsianskii

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INTEGRAL METHODS IN THE THEORY OF THE BOUNDARY LAYER*

By L. G. Loitsianskii

SUMMARY

The application of the well-known basic principle of mechanics, the principle of Jourdain, to problems of the theory of the boundary layer leads to an equation from which the equations of Von Karman, Leibenson, and Golubev are derived as special cases. The given equation may be employed in other integral methods. The present paper deals with the method of the variation of the thickness of the boundary layer. A number of new approximate formulas valuable in aerodynamic calculations for the friction distribution are derived from this procedure. The method has been applied only to laminar boundary layers, but it seems probable that it may be generalized to include turbulent layers as well.

The first step in the field of application of integral methods to the theory of the boundary layer was made by Von Karman in 1921. The fundamental idea of the Von Karman method (reference 1) consists in replacing the true velocity distribution in the boundary layer by certain approximate velocity distributions satisfying the differential equations of the boundary layer only on the average. For the averaging method Von Karman applied the usual integral mean over the layer, a procedure which led to a certain integral condition which is no other than the theorem on the change of momentum of a fluid in a volume element of the boundary layer. In the application of this method fundamental significance is attached, of course, to the choice of the system of functions that replaced the true velocity distribution, a circumstance analogous to the corresponding methods of the theory of elasticity (methods of Ritz, Galerkin, and others) and presenting, in general, great difficulties in using the so-called "direct" methods.

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In attempting to employ polynomial approximations for the velocity distribution, Von Karman found it was necessary to dispense with the theory of the boundary layer of infinite thickness (asymptotic approach of the velocity in the layer to the velocity of the outer flow) and introduce the concept of the layer of finite thickness. If, instead of polynomials, transcendental functions are applied as the approximating functions, the usual assumption concerning the asymptotic boundary layer can be used.

The method of Von Karman came into general use and was successfully applied in practice. At the same time, however, a number of investigators have pointed out certain defects: namely, the restricted field of application and the absence of any real basis (reference 2). The existence of a single integral condition permits making use of a family of functions approximating the velocity profile in the layer with the functions having only one parameter: namely, the thickness of the boundary layer. The approximation is obtained by making use of the boundary conditions at the wall and at the edge of the layer. The number of boundary conditions, both those given in advance and those derived from the equations of motion, is very limited; and, moreover, among these conditions the value of the first derivative of the velocity along the normal to the surface of the body does not enter. This derivative is precisely the fundamental unknown proportional to the frictional stress at a given point of the body. The choice of any particular boundary conditions leads to "external" solutions more accurately describing the phenomena near the outside edge of the layer or to "internal" solutions approximating the flow conditions near the walls. (See reference 3.) The well-known solution of Pohlhausen (reference 4) is an external solution. In setting up the approximating polynomial of the fourth degree three conditions were employed (the values of the function and of the first and second derivatives) at the outer edge and only two at the wall, the first derivative being of necessity omitted from the other two conditions, only the values of the function and of its second derivative being used. The Pohlhausen solution gives an exaggerated friction and a retarded boundary layer separation. At the author's suggestion, A. N. Alexandrov (reference 5) applied a second system of conditions leading to an internal solution, a reduced friction, and a somewhat earlier separation. By making use of a polynomial of the sixth degree a certain added accuracy could have been obtained.

Further progress in the development of integral methods was made by setting up equations which were a generalization of the momentum equation given by Von Karman. L. S. Leibenson (reference 6) proposed as a second integral condition the equation of the change of the kinetic energy of an element of the boundary layer, the condition being obtained by integration of both sides of the Prandtl equation, first multiplied by the forward velocity along the normal to the wall from the surface of the body to the edge of the layer. V. V. Golubev (reference 7) pointed out the infinite possibility of other integral conditions obtained by the same device of multiplying the two sides of the Prandtl equation by successive integral powers of the forward velocity. The same method for the particular case of a flat plate was followed by W. Sutton (reference 8). None of these investigators made any attempt to indicate methods for making use of the derived new integral conditions. Only W. Sutton gave an example of the computation of the friction of a plate, making use of a set of two equations. It is not clear, however, whether any of the proposed conditions could be used individually or whether it was necessary to use two of them combined.

In the present paper use is made of a single general principle of mechanics which, as far as is known, has not yet been applied in hydrodynamics and, in analytical mechanics bears the name of the Jourdain principle (reference 9). This principle, which is intermediate between the well-known principles of D'Alembert and Gauss, is very convenient for problems of hydrodynamics in the applications of Euler and, in particular, for obtaining "direct" methods of the solution of boundary layer problems.

The assigning of a particular form for the velocity variation in the expression of the Jourdain principle leads to the equation of Von Kármán, the system of equations of Leibenson-Golubev, and other methods which may be useful in applications.

1. The principle of Jourdain is obtained from the principle of D'Alembert by differentiation with respect to time. It is convenient to take the general equation of mechanics in the form:

$$\sum_{i=1}^n (m_i \dot{w}_i - F_i) \delta r_i = 0 \quad (1.1)$$

where m_i , w_i , and F_i are, respectively, mass, acceleration, and given force for a certain i th point and δr_i is a virtual displacement. By differentiating with respect to time, there is obtained:

$$\sum_{i=1}^n (m_i w_i - F_i) \frac{d\delta r_i}{dt} + \sum_{i=1}^n \frac{d}{dt} (m_i w_i - F_i) \delta r_i = 0 \quad (1.2)$$

By the definition of virtual displacements:

$$\frac{d}{dt} \delta r_i = \delta \dot{r}_i = \delta v_i$$

where v_i is the velocity of the i th point.

Let the actual motion of the system at a given instant be compared with a neighboring motion differing from the true one in the velocities but not in the positions — that is, let $\delta r_i = 0$. Equation (1.2) then assumes the form:

$$\sum_{i=1}^n (m_i w_i - F_i) \delta v_i = 0 \quad (1.3)$$

This is the general equation of the principle of Jourdain. It is to be emphasized that F_i are given forces while the reactions of the ideal connections N_i satisfying the condition

$$\sum_{i=1}^n N_i \delta r_i = 0$$

do not enter equation (1.3), since by the foregoing explanation

$$\sum_{i=1}^n N_i \delta v_i = 0$$

The frictional forces, as always, belong to the class of given forces.

In applying the Jourdain principle to the motion of a viscous incompressible fluid the equation of the prin-

ciple may be written in the following most general form:

$$\int_{\tau} (\rho w - \rho F + \text{grad } p - \mu \nabla^2 v) \delta v \, d\tau = 0 \quad (1.4)$$

where τ is an arbitrary volume of the fluid bounded by the surface σ ; δv is not subject to any restrictions except the condition of the existence of the integral on the left side of the equation.

It may be noted that, on the part of the surface σ corresponding to a solid wall, the variation δv should not, in advance, be subject to the condition $(\delta v)_n = 0$ since normal pressures enter the expression for the principle. In the same way, from the condition of the adherence of the fluid to the wall it does not follow that δv must be set equal to zero because in the equation (1.4) the frictional forces are taken into account.

If it is assumed that all the forces, both given and reactions, constitute an independent system, the fluid may be regarded as a free system of points not subject to any relations. The condition of the conservation of mass for a constant density $\text{div } v = 0$ likewise imposes no restrictions on the variations of the velocity. Various particular principles and theorems of mechanics will be obtained by restricting the variation of the velocity. Thus, for example, the momentum theorem is obtained by setting $\delta v = \epsilon$, where ϵ is an infinitely small vector independent of the coordinates; the theorem on the change in the kinetic energy is obtained by setting $\delta v = v\epsilon$ where ϵ is an infinitely small scalar independent of the coordinates; and so forth.

Considering only the case of steady motion and volumetric forces having a potential Π the principle is rewritten in the form

$$\int_{\tau} (\text{rot } v \times v + \text{grad } B + \nu \text{rot rot } v) \delta v \, d\tau = 0 \quad (1.5)$$

where

$$B = \frac{v^2}{2} + \Pi + \frac{p}{\rho}$$

whence by simple vector transformations there is obtained

$$\int_{\tau} (\text{rot } \mathbf{v} \times \mathbf{v}) \delta \mathbf{v} \, d\tau + \int_{\tau} \text{div}(\mathbf{B} \delta \mathbf{v}) \, d\tau - \int_{\tau} \mathbf{B} \, \text{div } \delta \mathbf{v} \, d\tau + \nu \int_{\tau} \text{div}(\text{rot } \mathbf{v} \times \delta \mathbf{v}) \, d\tau + \nu \int_{\tau} \text{rot } \mathbf{v} \times \delta \, \text{rot } \mathbf{v} \, d\tau = 0 \quad (1.6)$$

In the second and fourth integrals, in passing from volume to surface integrals, there is obtained

$$\int_{\tau} (\text{rot } \mathbf{v} \times \mathbf{v}) \delta \mathbf{v} \, d\tau + \int_{\sigma} \mathbf{B} \delta \mathbf{v}_n \, d\sigma - \int_{\tau} \mathbf{B} \, \text{div } \delta \mathbf{v} \, d\tau + \nu \int_{\sigma} (\text{rot } \mathbf{v} \times \delta \mathbf{v})_n \, d\sigma + \frac{1}{2} \nu \delta \int_{\tau} \text{rot } \mathbf{v} \times \text{rot } \mathbf{v} \, d\tau = 0 \quad (1.7)$$

The velocity variation $\delta \mathbf{v}$ now is subjected to the two conditions:

1. The neighboring motion satisfies the condition of incompressibility $\text{div } \delta \mathbf{v} = 0$.
2. On the boundary surface σ the velocity variation $\delta \mathbf{v}$ is equal to zero.

Under these assumptions the fundamental equation (1.7) assumes the form:

$$\int_{\tau} (\text{rot } \mathbf{v} \times \mathbf{v}) \delta \mathbf{v} \, d\tau + \frac{\nu}{2} \delta \int_{\tau} \text{rot } \mathbf{v} \times \text{rot } \mathbf{v} \, d\tau = 0 \quad (1.8)$$

This is the variational form of the principle of Jourdain in hydrodynamics for very general restrictions imposed on the velocity variation.

For particular cases of motion the principle takes a still simpler form. Consider, for example, the steady motion of a viscous, incompressible fluid at very small Reynolds numbers. The inertia terms in the hydrodynamic equations may be neglected. In equation (1.8) the first integral drops out and this leaves

$$\delta \int_T \text{rot } \mathbf{v} \times \text{rot } \mathbf{v} \, d\tau = 0 \quad (1.9)$$

which is no other than the known principle of the minimum dissipated energy first enunciated by Helmholtz and Rayleigh. Equation (1.9) maintains the same simple form also for any Reynolds numbers if the motion occurs with vortices parallel to the velocity ("free" vortices of the wing).

If equation (1.8) is rewritten in the equivalent form

$$\int_T (\mathbf{v} \times \delta \mathbf{v}) \text{rot } \mathbf{v} \, d\tau + \frac{1}{2} \mathbf{v} \delta \int_T \text{rot } \mathbf{v} \times \text{rot } \mathbf{v} \, d\tau = 0 \quad (1.10)$$

it is noted that, by subjecting the velocity variation to still another restriction,

$$\mathbf{v} \times \delta \mathbf{v} = 0 \quad (1.11)$$

that is, if it is required that the varied motion have the same direction of the velocities at a given instant as the true velocities, there is again obtained the principle in the simple form (1.9). The general form of the velocity variation now will be

$$\delta \mathbf{v} = \mathbf{v} \epsilon(x, y, z) \quad (1.12)$$

where the infinitely small function $\epsilon(x, y, z)$ is arbitrary and is subject only to the restriction

$$\text{div } \delta \mathbf{v} = \epsilon \text{div } \mathbf{v} + \mathbf{v} \times \text{grad } \epsilon = 0$$

which, for an incompressible fluid, is transformed into the condition:

$$\mathbf{v} \times \text{grad } \epsilon = 0 \quad (1.13)$$

This condition has a simple meaning: namely, the streamlines should lie on the level surfaces of the function ϵ . In two-dimensional motion with axial symmetry this simply means that ϵ should be an arbitrary function of the stream function ψ — that is,

$$\delta v = v \epsilon(\psi) \quad (1.14)$$

The condition that the velocity variation become zero at the boundary σ reduces to the condition that the closed surface σ should be a surface of flow.

In application, it is found simpler to make use of the principle in its general form (1.4), since the imposing of the above conditions on the velocity variation involves great difficulties. Application of the principle in the boundary layer theory will now be considered.

2. The case of a two-dimensional steady laminar boundary layer of finite thickness.— The usual considerations of the Prandtl theory with regard to the relative smallness, for large Reynolds numbers, of the transverse lengths and velocities lead to the principle of Jourdain in the form

$$\int_{\sigma} \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{dp}{dx} - v \frac{\partial^2 u}{\partial y^2} \right) \delta u \, d\sigma = 0 \quad (2.1)$$

where σ is an arbitrary region of the boundary layer. Choosing for the region σ the part of the layer bounded by the straight lines $x = x_1$, $x = x_2$, the wall of the body and the outer edge of the layer $y = h(x)$ gives

$$\int_{x_1}^{x_2} dx \int_0^{h(x)} \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{dp}{dx} - v \frac{\partial^2 u}{\partial y^2} \right) \delta u \, dy = 0 \quad (2.2)$$

If δu does not depend on the choice of x_1 and x_2 , then from the condition of arbitrariness of x_1 and x_2 the following equation will be obtained:

$$\int_0^{h(x)} \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{dp}{dx} - v \frac{\partial^2 u}{\partial y^2} \right) \delta u \, dy = 0 \quad (2.3)$$

It is seen readily that for various values of δu different integral conditions will be obtained including the above-mentioned integral conditions of Von Kármán, Leibenson, and Golubev. Thus, for example, setting

$$\delta u = \epsilon = \text{constant}$$

yields, after integrating, the equation of Von Karmán:

$$\frac{d}{dx} \int_0^h \rho u^2 dy - U \frac{d}{dx} \int_0^h \rho u dy = -h \frac{dp}{dx} - \mu \left(\frac{\partial u}{\partial y} \right)_{y=0} \quad (2.4)$$

where $U(x)$ is the velocity at the outer edge of the layer. In the same manner, setting (ϵ is an arbitrary, infinitely small quantity not depending on the coordinates)

$$\delta u = \epsilon u^k$$

gives the system of integral conditions of Golubev:

$$\int_0^h \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{dp}{dx} - v \frac{\partial^2 u}{\partial y^2} \right) u^k dy = 0 \quad (2.5)$$

$$(k = 0, 1, 2, \dots, \infty)$$

or, after some transformations:

$$\begin{aligned} & \frac{d}{dx} \int_0^h \frac{u^{k+2}}{k+1} dy - \frac{U^{k+1}}{k+1} \frac{d}{dx} \int_0^h u dy \\ &= - \frac{1}{\rho} \frac{dp}{dx} \int_0^h u^k dy - v_k \int_0^h u^{k-1} \left(\frac{\partial u}{\partial y} \right)^2 dy \end{aligned} \quad (2.6)$$

For $k = 1$, there is the equation of Leibenson. In deriving the system of equations (2.6), use was made of the arbitrariness of the variation δu .

If in the system of equations (2.5) the author passes from the variables x and y to the variables x and $\xi = u/U(x)$, which, in the region of the layer up to the separation point, are in a reciprocal 1:1 relation, the system of equations thus obtained:

$$\int_0^1 L(x, \xi) \xi^k d\xi = 0 \quad (k = 0, 1, 2, \dots, \infty)$$

may be considered as the conditions of setting equal to zero all the moments of the continuous function $L(x, \xi)$. Thus if possible to find a velocity distribution in the boundary layer which satisfies the infinite system of equation (2.5) or (2.6) such profile would also satisfy the Prandtl equation—that is, would give an exact solution of the problem of the boundary layer.

The question as to convergence of the method in using a finite number of equation (2.5) requires, of course, independent investigation in each case. If such convergence exists the condition of Von Kármán would be the equation of the first approximation, the combination of the equations of Von Kármán and Leibenson would correspond to the second approximation, and so forth.

The equation (2.3) may be considered as the integral condition corresponding to the general principle of Jourdain. In this equation the velocity variation plays the part of the "weight" of the integral mean. In the integral method of Von Kármán the weight is equal to unity, each element of the integral being assigned the same weight. In the Leibenson equation the velocity u serves as the weight. Since the velocity u becomes zero at the wall and assumes the maximum value at the edge of the layer it may be expected that the velocity profile satisfying the single condition of Leibenson will approach nearer the true value at the outer region of the layer than near the wall. The same is true with regard to the conditions of Golubev for $k = 1$.

Wide use may be made of the arbitrariness in the choice of the variation δu for the purpose of obtaining any particular weight of the integral mean.

The equations of Leibenson-Golubev, written in the form (2.5) and (2.6), as the equation of Von Kármán in the form (2.4), may be applied only to a layer with finite thickness. In using the more accurate theory of the asymptotic layer these equations lose their validity because the integrals become infinite. Another more convenient form of the equations may be indicated, valid for both the finite and the asymptotic layers.

To derive these equations it is necessary to pass, in the equations of the boundary layer, from the velocity u to the so-called "velocity deficiency" $q = U - u$ where U is the velocity at the outer edge assumed in advance as a given function of x . Use is made of the continuity equation to obtain

$$- 2U' q - U \frac{\partial q}{\partial x} + \frac{\partial}{\partial x} (q^2) - \frac{\partial}{\partial y} (qv) = - v \frac{\partial^2 q}{\partial y^2}$$

Multiplying both sides by q^k and integrating with respect to y between the limits zero to the thickness of the boundary h or to infinity in both cases after simple transformations and integration by parts there is obtained:

$$\begin{aligned} \frac{d}{dx} \left[U^{k+2} \int_0^{\infty, h} \left(\frac{q}{U} \right)^{k+1} \left(1 - \frac{q}{U} \right) dy \right] + (k+1) U^{k+1} U' \int_0^{\infty, h} \left(\frac{q}{U} \right)^{k+1} dy \\ = (k+1) v \int_0^{\infty, h} q^k \frac{\partial^2 q}{\partial y^2} dy \end{aligned} \quad (2.7)$$

By analogy with the so-called "displacement thickness" δ^* and "momentum thickness" δ^{**} equal, respectively, to

$$\begin{aligned} \delta^* &= \int_0^{\infty, h} \left(1 - \frac{u}{U} \right) dy = \int_0^{\infty, h} \frac{q}{U} dy \\ \delta^{**} &= \int_0^{\infty, h} \frac{u}{U} \left(1 - \frac{u}{U} \right) dy = \int_0^{\infty, h} \frac{q}{U} \left(1 - \frac{q}{U} \right) dy \end{aligned}$$

are introduced the thicknesses

$$\begin{aligned} h_k^* &= \int_0^{\infty, h} \left(\frac{q}{U} \right)^k dy = \int_0^{\infty, h} \left(1 - \frac{u}{U} \right)^k dy \\ h_k^{**} &= \int_0^{\infty, h} \left(\frac{q}{U} \right)^k \left(1 - \frac{q}{U} \right) dy = \int_0^{\infty, h} \left(1 - \frac{u}{U} \right)^k \frac{u}{U} dy \end{aligned} \quad (h_k^{**} = h_k^* - h_{k+1}^*)$$

Equation (2.7) then assumes the form

$$U^{k+2} \frac{dh_{k+1}^{**}}{dx} + (k+2) U^{k+1} U' h_{k+1}^{**} + (k+1) U^{k+1} U' h_{k+1}^* \\ = (k+1) \nu \int_0^\infty q^k \frac{\partial^2 q}{\partial y^2} dy$$

Dividing both sides of this equation by U^{k+2} finally gives:

$$\frac{dh_{k+1}^{**}}{dx} + \frac{U'}{U} \left[(k+2) h_{k+1}^{**} + (k+1) h_{k+1}^* \right] \\ = - (k+1) \frac{\nu}{U} \int_0^{\infty, h} \left(1 - \frac{u}{U} \right)^k \frac{\partial^2}{\partial y^2} \left(\frac{u}{U} \right) dy \quad (2.8)$$

This system of equations corresponds to the general equation (2.3) for $\delta u = (U-u)^k \epsilon = q^k \epsilon$ and, in contrast with the equations of Leibenson-Golubev, may be applied both to the layer of finite thickness and the asymptotic layer.

For $k = 0$ the thicknesses h_1^* and h_1^{**} become the usual displacement thickness δ^* and momentum thickness δ^{**} and equation (2.8) passes over into the equation of Von Karman in the form (In this paper to denote the thickness of the layer the letter h , and not δ , is used so as not to confuse this symbol with the variation sign. In equation (2.9) the usual notation is kept.) which is now best taken as:

$$\frac{d\delta^{**}}{dx} + \frac{U'}{U} (2\delta^{**} + \delta^*) = \frac{\tau_0}{\rho U^2} \quad (2.9)$$

where τ_0 is the friction stress at the wall:

$$\tau_0 = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0} \quad (2.10)$$

Equations (2.9) and (2.10) may be considered, respectively, as the integral and differential (with respect to

the coordinate y) determinations of the frictional stresses for a given velocity distribution $u(x, y)$. The approximate differentiation of a given function $u(x, y)$ is a very inaccurate operation for determining the friction. In general, the integral method of determination (2.9) is to be preferred. The only exception may be the region where the velocity profiles rapidly change their shape and inaccuracy in determining the derivative ds^{**}/dx may show up. This will be the case in the region near the separation of the boundary layer. Thus, in determining the local frictional stresses, it is recommended to use the integral method (2.9) everywhere except in the region near the separation. For determining the separation point, however, it is necessary to obtain internal solutions — that is, solutions which are nearer the true values at the walls than at the edge of the layer and at the separation point equate to zero the friction determined by the differential method.

From this point of view the Von Kármán method, based on the application of equation (2.9), in which the right side is replaced by (2.10), is no other than the condition that the local frictional stress determined by the differential and integral methods be the same. This condition, in using only a single parameter (thickness of the boundary layer), is difficult to satisfy even by imposing on the velocity profile a large number of boundary conditions because by using this condition use is not made of the specific advantages of the differential and integral expressions for the friction. But by use of a method of approximate determination the velocity profile that does not involve the use of the Von Karman equation it is possible to apply equation (2.9) as the integral expression of the local frictional stress.

These considerations apply entirely to the Leibenson-Golubev equations, since the Von Karman equation is included and is fundamental in the system of successive approximations (setting the moment of zero order equal to zero).

As usual in the practical applications of direct methods, only the first approximation is of significance, since to obtain the second and following approximations is a problem of great difficulty. It will be shown how, by making use of the general equation (2.3) and the integral expression for the friction, it is possible to obtain a first approximation more accurate than the solution of Von Karman-Pohlhausen.

3. It is best to start with the simplest case of the flow about a plate. Following the idea of a boundary layer

of finite thickness, the velocity distribution in a cross-section of the layer is expressed in the form:

$$u = U g_0(\eta) = U g_0(y/h) \quad (3.1)$$

where U = constant is the velocity of the approaching flow and $g_0(\eta)$ is a function satisfying some condition on the surface of the plate ($\eta = 0$) and at the outer edge of the layer ($\eta = 1$). If the thickness of the layer $h(x)$ is the fundamental and only parameter in the velocity distribution, the velocity variation will be set up by varying this parameter. Then

$$\delta u = U g_0'(\eta) \delta \eta = - U \eta g_0'(\eta) \frac{\delta h}{h} \quad (3.2)$$

Simple computations give:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \nu \frac{\partial^2 u}{\partial y^2} = - U^2 \frac{h'}{h} g_0' \int_0^\eta g_0 d\eta - \nu U \frac{g_0''}{h^2}$$

Substituting these expressions in the general equation (2.3) yields, because of the arbitrariness of δh :

$$- U h' \int_0^1 \eta g_0'^2 \left(\int_0^\eta g_0 d\eta \right) d\eta - \frac{\nu}{h} \int_0^1 \eta g_0' g_0'' d\eta = 0$$

On the variation $\delta \eta$ is imposed the condition of becoming equal to zero on the boundary of the interval $\eta = 0, \eta = 1$; for this by (3.2) it is required that $g_0'(1) = 0$. Then integrating by parts the second integral in the above equation gives

$$\frac{d}{dx} h^2 = \alpha^2 \frac{\nu}{U}$$

where

$$\alpha^2 = \left\{ \int_0^1 g_0'^2 d\eta \right\} : \left\{ \int_0^1 \eta g_0'^2 \left(\int_0^\eta g_0 d\eta \right) d\eta \right\} = \text{const.} \quad (3.3)$$

For the condition $h(0) = 0$ there is obtained:

$$h = \alpha \sqrt{\nu x / U} \quad (3.4)$$

The two equations (3.4) and (3.3) constitute a solution of the problem of the thickness of the boundary layer.

The local frictional stress may be determined by the differentiation method by the formula:

$$\tau_d = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0} = \mu \frac{U}{h} g_0'(0) = \frac{g_0'(0)}{\alpha} \sqrt{\frac{\mu \rho U^3}{x}} \quad (3.5)$$

or by the integral method on the basis of (2.9);

$$\tau_i = \rho U^2 \frac{d\delta^{**}}{dx} = \left[\rho U^2 \int_0^1 g_0(1 - g_0) d\eta \right] \frac{dh}{dx} = \frac{1}{2} \alpha \beta \sqrt{\frac{\mu \rho U^3}{x}} \quad (3.6)$$

where β denotes the constant:

$$\beta = \int_0^1 g_0(1 - g_0) d\eta \quad (3.7)$$

In the assumed averaging the weight δu becomes zero at the wall and at the outside boundary of the layer. It should be expected here that the velocity profile obtained will approach the true one in the middle part of the boundary layer and give sufficiently accurate values for the friction determined integrally by equation (3.6)

The application of the simple parabolic distribution

$$g_0 = 2\eta - \eta^2$$

satisfying the condition $u = 0$ for $y = 0$, $u = U$ and $\partial u / \partial y = 0$ for $y = h(x)$ gives, as simple computations show, the friction formula

$$\tau_i = 0.331 \sqrt{\mu \rho U^3 / x}$$

differing by only 0.3 percent of the accurate formula of Blasius, while application of the same distribution in the Von Karman method leads to the formula:

$$\tau_0 = 0.365 \sqrt{\mu \rho U^3 / x}$$

exceeding the exact solution by 10 percent

TABLE 1

$g_0(\eta)$	α	$\tau_d \sqrt{\frac{x}{\mu \rho U^3}}$	$\tau_i \sqrt{\frac{x}{\mu \rho U^3}}$	$\tau_0 \sqrt{\frac{x}{\mu \rho U^3}}$
$3\eta - 3\eta^2 + \eta^3$	3.25	0.92	0.17	0.39
$\frac{3}{2}\eta - \frac{1}{2}\eta^3$	3.6	.42	.25	0.32
$\frac{4}{3}\eta - \frac{1}{3}\eta^4$	3.8	.35	.27	0.31
$\sin \frac{\pi\eta}{2}$	4.42	.36	.31	0.33
$2\eta - \eta^2$	4.95	.40	.331	0.36
$2\eta - 2\eta^3 + \eta^4$	5.62	.36	.331	0.34
Blasius' exact solution	—	.332	.332	—

If $g_0(\eta)$ is subjected to a larger number of conditions by making use of the values of the second and third derivatives at the wall and edge of the layer, another series of values of the coefficients in the integral formula for the friction may be obtained. In table 1 the results of computations arranged in the order of increasing thickness of the layer (coefficient α) are given. As may be seen from the second and fourth columns, with increasing thickness of the layer (coefficient α) the friction determined integrally at first increases to a value near the true one and then decreases. In the last column

are given the values of the friction coefficients according to Pohlhausen for which the rule just given does not hold. It may be easily shown that in the particular case of a plate the friction coefficients according to Pohlhausen are the geometric mean of our differential and integral coefficients.

4. It is not difficult to generalize the above device of varying the thickness of the layer to the case of two-dimensional flow about an arbitrary body.

It is well to start with the simplest example of a parabolic approximation of the velocity profile in the boundary layer. This, as has been shown, immediately leads to a very simple formula for the solution in finite form of the friction problem. In contrast to the plate where the velocity of the outside flow is the same throughout, in the case of an arbitrary body, the velocity U in the formula

$$u = U g_0(\eta) = U(2\eta - \eta^2) \quad (4.1)$$

will be a given function $U(x)$ of the abscissa x ,

The value of the variation of the velocity δu remains the same as in the previous section, but the expression in parentheses in the fundamental equation (2.3) will become somewhat more complicated after substituting (4.1). This expression will be given later in a general form, but for the present it is enough to show that, in the concrete case, (4.1), after very simple computations, equation (2.3) is reduced to the differential equation:

$$\frac{17}{630} \frac{h'}{h} + \frac{13}{126} \frac{U'}{U} - \frac{v}{3h^2U} = 0 \quad (4.2)$$

With introduction, as is customary, of the function $z = h^2/v$ equation (4.2) assumes the form

$$\frac{dz}{dx} = 7.65 \frac{U'}{U} z = 24.7 \frac{1}{U} \quad (4.3)$$

If the above equation is rewritten in the form

$$\frac{dz}{dx} = \frac{24.7 - 7.65 U'z}{U}$$

it is seen that if $U(0) = 0$ for $x = 0$ the point $x = 0$ is a singular point of the equation. Imposing on the derivative dz/dx the condition that it be finite for $x = 0$ results in the equation:

$$24.7 - 7.65 U_0' z_0 = 0$$

for determining the initial value of z_0 for $x = 0$. From this equation it follows that the initial value of the parameter of Pohlhausen λ in the present case is equal to

$$\lambda_0 = U_0' z_0 = 3.23 \quad (4.4)$$

Thus to the equation (2.3) there is added the initial condition

$$z = 3.23/U_0' \text{ for } x = 0$$

Equation (4.3) is linear and has the solution:

$$z = \frac{h^2}{v} = 24.7 U^{-7.65} \int_0^x U^{6.65} dx \quad (4.5)$$

If, as in the case of the plate, $U \neq 0$ for $x = 0$ the initial value of z is equal to zero — that is, the initial thickness of the layer is equal to zero.

Again making use of the integral determination of the friction, equation (2.9) results in ($\delta^{**} = 2h/15$, $\delta^* = h/3$):

$$\frac{\tau_1}{\rho U^2} = \frac{0.331 U^{0.825} \sqrt{v}}{\sqrt{\int_0^x U^{6.65} dx}} \left(1 + 1.35 U' U^{-7.65} \int_0^x U^{6.65} dx \right) \quad (4.6)$$

In the case of the plate ($U' = 0$) formula (4.6) reduces to the formula previously obtained for the friction at the

plate. (The computations in the given example for the parabolic approximation were carried out by Stepaniantz. The latter also gives a generalization of formula (4.6) for the case of a flow with axial symmetry about a body of revolution.)

Equation (4.6) gives a sufficiently accurate determination of the friction in that part of the boundary layer where U' is near zero. As has already been pointed out, the accuracy for the plate exceeds 0.3 percent. Near the forward stagnation point, however, and at the separation point where $U' > 0$ or $U' < 0$ the accuracy drops, the friction obtained is too low at the forward stagnation point and too large near the separation point. Since in practice in the diffuser part a transition occurs from the laminar to the turbulent layer, this defect is not of great significance. Formula (4.6) may be recommended for practical computations in view of its great simplicity. For the function $U(x)$ graphically given the integrals may be easily computed with the aid of mechanical devices while the derivative $U'(x)$ is determined by graphical differentiation. The use of the method of Von Karman-Pohlhausen for a parabolic velocity profile also leads to a simple friction formula but one of very small accuracy (for a plate the error, as has already been pointed out, is equal to 10 percent); in the diffuser region it gives a greatly exaggerated friction and no separation at all while formula (4.6) gives a separation point though somewhat retarded.

For a comparative estimate of formula (4.6) with the formula of Von Karman-Pohlhausen near the forward critical point $x = 0$, $U = 0$ the Von Karman-Pohlhausen formula is given:

$$\frac{\tau_0}{\rho U^2} = \frac{0.365 U^{3.5} \sqrt{v}}{\int_0^x U^8 dx} \quad (4.7)$$

In the immediate neighborhood of the point $x = 0$ the velocity of the external flow will be represented in the form $U = cx$. Then, as simple computation shows, formula (4.6) gives:

$$\frac{\tau_1}{\rho U^2} = 1.08 \sqrt{\frac{v}{c}} \frac{1}{x} \quad (4.8)$$

while formula (4.7) leads to the expression

$$\frac{\tau_0}{\rho U^2} = 1.10 \sqrt{\frac{v}{c}} \frac{1}{x} \quad (4.8')$$

In the simple case considered $U = cx$ the accurate solution may also be given (reference 10):

$$\frac{\tau_0}{\rho U^2} = 1.23264 \sqrt{\frac{v}{c}} \frac{1}{x} \quad (4.9)$$

It is seen that in the immediate neighborhood of the critical point both comparison formulas give lowered solutions while the order of error is about 10 percent. When $U(x)$ deviates from the straight line $U = cx$ formula (4.6) gives a solution closer to the true one than formula (4.7), which gives too great a value for the friction. Present velocity profiles for flight angles of attack are such that only very near to $x = 0$ is there a region for which a straight line $U = cx$ approximately holds true. Even for very small values of x the curve rapidly passes over into a region of almost constant value of the velocity U and therefore small U' . For such a type of curve formula (4.6) should give sufficiently accurate values for the friction.

In order to estimate the accuracy of formula (4.6) in the region beyond the maximum velocity (diffuser region), consider the example of the profile

$$U = b_0 - b_1 x \quad (4.10)$$

representing the simplest linear drop of the velocity of the external flow beyond the maximum point. In equation (4.10) the coefficient b_0 gives the maximum value of the velocity U_0 for $x = 0$ and $b_1 = -U' = \text{constant}$, the slope of the velocity curve. Substituting for x a new variable $x^* = b_1 x / b_0$ gives

$$\begin{aligned} \int_0^x U^{6.65} dx &= \frac{b_0^{7.65}}{b_1} \int_0^{x^*} (1-x^*)^{6.65} dx^* \\ &= \frac{b_0^{7.65}}{7.65 b_1} [1 - (1-x^*)^{7.65}] \end{aligned}$$

Substituting the value of this integral in formula (4.6) results in

$$\frac{\tau_i}{\rho U^2} = \frac{0.331}{\sqrt{7.65}} \frac{\sqrt{b_1^2}}{b_0} \frac{(1-x^*)^{2.825}}{\sqrt{1-(1-x^*)^{7.65}}} [9 - 1.35(1-x^*)^{-7.65}]$$

Let the following magnitude independent of b_0 and b_1 be considered:

$$\begin{aligned} \frac{\tau_{i1}}{1-x^*} &= \frac{\tau_i}{b_0 \sqrt{\mu \rho b_1}} \frac{1}{1-x^*} \\ &= \frac{0.331}{\sqrt{7.65}} \frac{(1-x^*)^{3.825}}{\sqrt{1-(1-x^*)^{7.65}}} [9 - 1.35(1-x^*)^{-7.65}] \quad (4.11) \end{aligned}$$

The magnitude $\tau_{i1}/(1-x^*)$ was obtained by Howarth (reference 11) as a result of an accurate solution of the problem for the velocity profile given. For comparison, in table 2, the values of this magnitude are given as computed by Howarth and by formula (4.11).

TABLE 2

x^*	According to Howarth	By formula (4.11)
	$\frac{\tau_{i1}}{1-x^*}$	$\frac{\tau_{i1}}{1-x^*}$
0.0000	∞	∞
.0125	2.773	2.78
.0250	1.817	1.82
.0500	1.064	1.21
.1000	.345	.645
.1200	.000	
0.22		.000

The table shows that only for small values of x^* — that is, only at the start of the diffuser part is there sufficiently good agreement between formula (4.6) and the accurate solution. As the point of separation is approached ($x^* = 0.12$), as has been pointed out earlier, the method becomes

increasingly inaccurate. For $x^* = 0.05$ the approximate solution exceeds the accurate by almost 15 percent. Setting the right side of (4.11) equal to zero the abscissa of the separation point $x_s^* = 22$ is found, while the accurate value is $x_s^* = 0.12$. The Pohlhausen formula (4.7) gives a greatly exaggerated friction and leads to the incorrect conclusion of the absence of separation. Thus for a value as small as $x^* = 0.0125$ the increased friction, according to Pohlhausen, constitutes about 25 percent.

In 1939 the new formula of Wright and Baily (reference 12) for the frictional stress appeared:

$$\frac{\tau_o}{\rho U^2} = \frac{0.332}{\sqrt{Ux/\nu}} \left(1 + 8.18 \frac{U'x}{U} \right) \quad (4.12)$$

This formula, attractive on account of its extreme simplicity, was derived by Wright and Bailey on the basis of an experimentally confirmed assumption of the possibility (for small divergence and convergence) of substituting in equation (2.9) for δ^*/δ^{**} and $d\delta^{**}/dx$ the corresponding values for the plate. Formula (4.12) is accurately true for a plate but contains an error that is immediately evident to the eye: namely, the friction is determined by the values U and U' at a given point and does not depend on the previous development of the layer. It is readily seen that at the forward part it is extremely inaccurate. Thus if, as before, $U = cx$ there is obtained, by equation (4.12),

$$\frac{\tau_o}{\rho U^2} = 3.04 \sqrt{\frac{\nu}{c}} \frac{1}{x}$$

a result more than twice the accurate value (4.9). The proposed formula (4.6) is almost as simple as the formula of Wright and Baily and at the same time is free from the above-mentioned defects.

More accurate approximations will now be considered.

5. As is known, the velocity profile satisfying the boundary conditions at the wall and at the outer edge of a layer of finite thickness must be represented in the form:

$$u = U(x) [g_0(\eta) + \lambda g_1(\eta)] \quad (5.1)$$

where $\eta = y/h(x)$ and λ the known parameter of Pohlhausen is given by

$$\lambda = U! h^2 / \nu \quad (5.2)$$

The functions $g_0(\eta)$ and $g_1(\eta)$ are two polynomials the degree and coefficients of which depend on the choice of the particular boundary conditions.

The general form of relation (5.1) is kept and the fundamental equation (2.3) is set up, where use will be made of the device of varying the thickness of the layer, or what amounts to the same thing, varying the parameter λ . This will give the following equation ($\dot{}$ dot above a letter denotes differentiation with respect to η and a dash denotes differentiation with respect to x):

$$\begin{aligned} \delta u &= U (\delta g_0 + \lambda \delta g_1 + g_1 \delta \lambda) \\ &= U (\dot{g}_0 + \lambda \dot{g}_1) \left(-\frac{1}{2} \eta \frac{\delta \lambda}{\lambda} \right) + U g_1 \delta \lambda \\ &= \frac{U}{\lambda} \left[\left(g_1 - \frac{1}{2} \eta \dot{g}_1 \right) \lambda - \frac{1}{2} \eta \dot{g}_0 \right] \delta \lambda \quad (5.3) \end{aligned}$$

The expression in parenthesis in equation (2.3') may be given in the form:

$$\begin{aligned} L(u, v) &= u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - UU' - v \frac{\partial u}{\partial y} = \\ &= U(g_0 + \lambda g_1) \left[U(g_0 + \lambda g_1) + U g_1 \lambda' - \frac{1}{2} \eta U \left(\frac{\lambda'}{\lambda} - \frac{U''}{U'} \right) (g_0 + \lambda g_1) \right] - \\ &- \left[U \left(\int_0^1 g_0 d\eta + \lambda \int_0^1 g_1 d\eta \right) + U \lambda' \int_0^1 g_1 d\eta - \frac{1}{2} U \left(\frac{\lambda'}{\lambda} - \frac{U''}{U'} \right) \eta (g_0 + \lambda g_1) + \right. \\ &+ \left. \frac{1}{2} U \left(\frac{\lambda'}{\lambda} - \frac{U''}{U'} \right) \left(\int_0^1 g_0 d\eta + \lambda \int_0^1 g_1 d\eta \right) \right] U(g_0 + \lambda g_1) - UU' - \frac{UU''}{\lambda} (g_0 + \lambda g_1). \end{aligned} \quad (5.4)$$

Removing parenthesis and making certain simplifications yields:

$$\begin{aligned} L(u, v) &= UU' \left(g_0^2 - g_0 \int_0^1 g_0 d\eta - \bar{g}_1 - 1 \right) + UU' \lambda \left(2g_0 g_1 - g_0 \int_0^1 g_1 d\eta - \bar{g}_1 \int_0^1 g_0 d\eta \right) - \\ &- \frac{UU''}{\lambda} \bar{g}_0 + UU' \lambda^2 \left(g_1^2 - g_1 \int_0^1 g_1 d\eta \right) + U^2 \lambda' \left(g_1 g_0 - g_0 \int_0^1 g_1 d\eta \right) + \\ &+ U^2 \lambda \lambda' \left(g_1^2 - g_1 \int_0^1 g_1 d\eta \right) - \frac{1}{2} U^2 \left(\frac{\lambda'}{\lambda} - \frac{U''}{U'} \right) g_0 \int_0^1 g_0 d\eta - \\ &- \frac{1}{2} U^2 \lambda \left(\frac{\lambda'}{\lambda} - \frac{U''}{U'} \right) \left(g_0 \int_0^1 g_1 d\eta + g_1 \int_0^1 g_0 d\eta \right) - \frac{1}{2} U^2 \lambda^2 \left(\frac{\lambda'}{\lambda} - \frac{U''}{U'} \right) g_1 \int_0^1 g_1 d\eta. \end{aligned} \quad (5.5)$$

Substitution of the obtained expression $L(u, v)$ and δu in equation (2.3) results in (for complete arbitrariness of $\delta \lambda$) a differential equation with respect to λ :

$$\frac{d\lambda}{dx} = \frac{U'}{U} \chi(\lambda) + \frac{U''}{U'} \psi(\lambda), \quad (5.6)$$

where

$$\chi(\lambda) = - \frac{a_0 + a_1 \lambda + a_2 \lambda^2 + a_3 \lambda^3 + a_4 \lambda^4}{a_5 + a_6 \lambda + a_7 \lambda^2 + a_8 \lambda^3}, \quad \psi(\lambda) = - \frac{a_9 \lambda + a_{10} \lambda^2 + a_{11} \lambda^3 + a_{12} \lambda^4}{a_5 + a_6 \lambda + a_7 \lambda^2 + a_8 \lambda^3}. \quad (5.7)$$

The coefficients a_0, a_1, \dots, a_{12} are constants depending on the form of the functions g_0 and g_1 - that is, on the choice of the boundary conditions. These constants are determined, respectively, as the integrals with respect to η from 0 to 1 of the following functions of η :

$$\begin{aligned} \bar{a}_0 &= \frac{1}{2} \eta g_0 \bar{g}_0, \\ a_1 &= - \frac{1}{2} \eta g_0 \left(g_0^2 - g_0 \int_0^1 g_0 d\eta - \bar{g}_1 - 1 \right) - \bar{g}_0 \left(g_1 - \frac{1}{2} \eta g_1 \right), \\ \bar{a}_2 &= - \frac{1}{2} \eta g_0 \left(2g_0 g_1 - g_1 \int_0^1 g_0 d\eta - g_0 \int_0^1 g_1 d\eta \right) + \\ &+ \left(g_1 - \frac{1}{2} \eta g_1 \right) \left(g_0^2 - g_0 \int_0^1 g_0 d\eta - \bar{g}_1 - 1 \right). \end{aligned}$$

$$\begin{aligned}
\bar{a}_2 &= -\frac{1}{2} \eta \dot{g}_0 \left(g_1^2 - \dot{g}_1 \int_0^1 g_1 d\eta \right) + \left(g_1 - \frac{1}{2} \eta \dot{g}_1 \right) \left(2g_0 g_1 - \dot{g}_1 \int_0^1 g_0 d\eta - \dot{g}_0 \int_0^1 g_1 d\eta \right), \\
\bar{a}_4 &= \left(g_1 - \frac{1}{2} \eta \dot{g}_1 \right) \left(g_1^2 - \dot{g}_1 \int_0^1 g_1 d\eta \right), \quad \bar{a}_5 = \frac{1}{4} \eta \dot{g}_0^2 \int_0^1 g_0 d\eta, \\
\bar{a}_6 &= -\frac{1}{2} \eta \dot{g}_0 \left(g_0 g_1 - \dot{g}_0 \int_0^1 g_1 d\eta \right) + \frac{1}{4} \eta \dot{g}_0 \left(\dot{g}_1 \int_0^1 g_0 d\eta + \dot{g}_0 \int_0^1 g_1 d\eta \right) - \\
&\quad - \frac{1}{2} \left(g_1 - \frac{1}{2} \eta \dot{g}_1 \right) \dot{g}_0 \int_0^1 g_0 d\eta, \\
\bar{a}_7 &= -\frac{1}{2} \eta \dot{g}_0 \left(g_1^2 - \dot{g}_1 \int_0^1 g_1 d\eta \right) + \left(g_1 - \frac{1}{2} \eta \dot{g}_1 \right) \left(g_0 g_1 - \dot{g}_0 \int_0^1 g_1 d\eta \right) + \\
&\quad + \frac{1}{4} \eta \dot{g}_0 \dot{g}_1 \int_0^1 g_1 d\eta - \frac{1}{2} \left(g_1 - \frac{1}{2} \eta \dot{g}_1 \right) \left(\dot{g}_1 \int_0^1 g_0 d\eta + \dot{g}_0 \int_0^1 g_1 d\eta \right), \\
\bar{a}_8 &= \left(g_1 - \frac{1}{2} \eta \dot{g}_1 \right) \left(g_1^2 - \dot{g}_1 \int_0^1 g_1 d\eta \right) - \frac{1}{2} \left(g_1 - \frac{1}{2} \eta \dot{g}_1 \right) \dot{g}_1 \int_0^1 g_1 d\eta, \\
\bar{a}_9 &= -\frac{1}{4} \eta \dot{g}_0^2 \int_0^1 g_0 d\eta, \\
\bar{a}_{10} &= -\frac{1}{4} \eta \dot{g}_0 \left(\dot{g}_1 \int_0^1 g_0 d\eta + \dot{g}_0 \int_0^1 g_1 d\eta \right) + \frac{1}{2} \left(g_1 - \frac{1}{2} \eta \dot{g}_1 \right) \dot{g}_0 \int_0^1 g_0 d\eta, \\
\bar{a}_{11} &= -\frac{1}{4} \eta \dot{g}_0 \dot{g}_1 \int_0^1 g_1 d\eta + \frac{1}{2} \left(g_1 - \frac{1}{2} \eta \dot{g}_1 \right) \left(\dot{g}_1 \int_0^1 g_0 d\eta + \dot{g}_0 \int_0^1 g_1 d\eta \right), \\
\bar{a}_{12} &= \frac{1}{2} \left(g_1 - \frac{1}{2} \eta \dot{g}_1 \right) \dot{g}_1 \int_0^1 g_1 d\eta.
\end{aligned}$$

At request of the author, A. P. Krol has made computations for the boundary conditions:

$$\begin{aligned}
u &= 0, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{UU'}{v} \quad \text{for } y=0, \\
u &= U, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for } y=h,
\end{aligned} \tag{5.8}$$

in satisfying which the functions $g_0(\eta)$ and $g_1(\eta)$ have the form:

$$g_0(\eta) = 2\eta - 2\eta^3 + \eta^4, \quad g_1(\eta) = (\eta - 3\eta^3 + 3\eta^5 - \eta^7)/6. \tag{5.9}$$

The values of the computed coefficients are given below:

$$\begin{array}{ll}
a_0 = -0.37142857, & a_7 = -0.00003205, \\
a_1 = 0.10224665, & a_8 = 0.00000047, \\
a_2 = -0.00806332, & a_9 = -0.01174048, \\
a_3 = 0.00013143, & a_{10} = 0.001027, \\
a_4 = 0.00000379, & a_{11} = 0.000007, \\
a_5 = 0.01174048, & a_{12} = -0.00000096, \\
a_6 = -0.00127521, &
\end{array}$$

The functions $x(\lambda)$ and $\psi(\lambda)$ may be tabulated in advance

Equation (5.6) has the singular point $x = 0$, $\lambda = \lambda_0$; the usual investigation shows that λ_0 is the least positive root of the numerator of $x(\lambda)$ and is equal to $\lambda_0 = 6.73$. The second singular point will be the point $x = x_m$, $\lambda = 0$ of the minimum pressure.

As for the problem of the integral determination of the friction τ at the wall, there is first found:

$$\delta^* = (0.300000 - 0.008333\lambda)h = a(\lambda)h \quad (5.10)$$

$$\delta^{**} = (0.117500 - 0.001058\lambda - 0.0001102\lambda^2)h = b(\lambda)h$$

Then τ is determined by formula (2.9):

$$\frac{\tau_i}{\rho U^2} = \frac{d}{dx} \left[b(\lambda) \sqrt{\frac{v\lambda}{U'}} \right] + \frac{U'}{U} [2b(\lambda) + a(\lambda)] \sqrt{\frac{v\lambda}{U'}} \quad (5.11)$$

If desired, the differentiation may be carried out in the first term on the right and the derivative of λ with respect to x from equation (5.6) substituted.

To estimate the relative accuracy of the method, it is necessary to proceed as before. It is already known that for small values of U' the integral formula (5.11) gives excellent agreement with the accurate solution of Blasius (difference about 0.3 percent, table 1, in which are given the values of τ_i for the plate).

The behavior of the friction near the forward critical point must be considered next. It is noted that $U' = c$, $U'' = 0$, $(d\lambda/dx)_{x=0} = 0$ for $U = cx$ and for sufficiently small x it may be assumed that neglecting powers higher than the first — $\lambda = \lambda_0 = 6.73$. Then by equation (5.11) there is obtained:

$$\frac{\tau_i}{\rho U^2} = \sqrt{\lambda_0} [2b(\lambda_0) + a(\lambda_0)] \sqrt{\frac{v}{c}} \frac{1}{x} = 1.18 \sqrt{\frac{v}{c}} \frac{1}{x} \quad (5.12)$$

a result approximately 4 percent less than the accurate (4.10). Thus the chosen polynomial of the fourth degree

satisfying conditions (5.8) determines the friction near the critical point with almost two-and-a-half times the accuracy of the previously considered polynomial of the second degree and in the remaining region (excluding the region near the separation point) gives the same accuracy (of the order of 0.3 percent). The same approximation, according to Pohlhausen, gives near the critical point:

$$\frac{\tau_o}{\rho U^2} = 1.20 \sqrt{\frac{\nu}{c}} \frac{1}{x} \quad (5.13)$$

that is, only 3 percent less accurate, while with increasing distance from the critical point the accuracy of the solution (5.11) considerably exceeds the accuracy of the Pohlhausen solution.

The method considered in the present and preceding sections of estimating the accuracy of the solution permits the conclusion that the order of the error in the forward part does not depend on the steepness with increasing $U(x)$; the absolute error of the proposed solutions will accumulate in the region of increasing $U(x)$ and will be smaller the smaller the abscissa interval where the velocity approaches its maximum value.

The computations conducted by A. P. Krol for the diffuser region showed that the friction computed by the method just discussed is much nearer the true value than that computed by the method of Pohlhausen and other methods. Only in the immediate neighborhood of the point of separation does the method fail. The reasons for this fact require special investigation.

6. The method presented still requires additional improvement and simplification. It may be mentioned that L. G. Stepaniantz generalized the method of varying the thickness of the layer to the case of a flow with axial symmetry about a body of revolution. The results obtained show that the method may be successfully applied to the practical computation of bodies of revolution (of airship form).

It is hoped that later on the method of choice of the most suitable forms of the velocity variations for a single and several varying parameters will be made more accurate. It is apparent that there are no objections in principle to the application of the method not only to the laminar but also to the turbulent boundary layer.

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